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ON THE DISTRIBUTION OF THE INTEGRATED SQUARE
OF THE ORNSTEIN-UHLENBECK PROCESS

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INTRODUCTION

Let $X(t)$, $t \geq 0$, be the Ornstein-Uhlenbeck process: the stationary, mean-zero Gaussian Markov process with covariance $\text{Cov}(X(t), X(s)) = (\frac{1}{2})\exp(-|t-s|)$. Let

$$Z(T) = \int_0^T X^2(t) dt. \quad (1)$$

In this paper we calculate the Laplace transform of the probability density function of $Z(T)$ using functional integral methods, invert this transform by expanding it in an infinite series, and study the asymptotic behavior of the density and distribution functions as $T \rightarrow \infty$. The functional integral approach makes possible a parallel analysis of the density and distribution functions of $Z(T)$ conditioned on the event $X(T) = 0$, which are of interest in certain applications (Dankel, to appear). The asymptotic limits are all closely related to the inverse Gaussian distribution with parameters $(T/2, T/2)$ (Johnson and Kotz, 1970a), whose density function is, to within a constant factor, the asymptotic form of the conditional density of $Z(T)$, but not of its unconditional density (see Theorem 3 below).

$Z(T)$ is an example of a quadratic form in normal variables, a subject that has been widely studied; see (Johnson and Kotz, 1970b) for a comprehensive survey of work up to 1970. The usual method of analysis expresses the characteristic function of the distribution as an infinite product involving the eigenvalues of the covariance operator. Using this method to study a process very similar to our $Z(T)$, Stepien (1958) recognized the infinite product as an entire function expressible in terms of elementary functions, and inverted the Fourier transform via contour

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integration and numerical integration. By contrast, our functional integral method bypasses calculation of eigenvalues and infinite products, yielding explicit finite formulas for transforms directly; moreover, the terms in our series for the inverse transforms (i.e., the conditional and unconditional probability distribution functions of $Z(T)$) are given as explicit expressions involving standard higher transcendental functions. The explicit form of these terms makes asymptotic analysis feasible.

The approximation of the probability density function of $Z(T)$ by an inverse Gaussian density was heuristically introduced by Grenander, Pollak, and Slepian (1959); however, they gave no definite conditions for the validity of this approximation. In Theorem 3 below, we show that the approximation is asymptotically exact to within a constant factor, but only if one conditions on the event $X(T) = 0$. As far as we are aware, the study of such conditional distributions of quadratic forms in normal variables has been a field of research largely unexplored.

We shall require a number of formulas for Laplace transforms and special functions; these are given, along with references, in the appendices.

TRANSFORMS

In the sequel $F_T(t)$ will denote the probability distribution function of $Z(T)$, and $f_T(t)$ the corresponding probability density function. We shall use the superscript "c" to denote quantities conditional on the event $X(T) = 0$; thus, $F_T^C(t)$ and $f_T^C(t)$ will denote the conditional distribution and density functions of $Z(T)$, respectively. We shall denote the Laplace transform of a function $f(t)$, $t \in [0, \infty)$, by $\mathcal{L}(f(t))(s)$.

Theorem 1:

$$(A) \mathcal{L}(f_T(t))(s)$$

$$= e^{T/2} \left[\frac{1}{2} \left[(1+2s)^{\frac{1}{2}} + (1+2s)^{-\frac{1}{2}} \right] \sinh \left[T(1+2s)^{\frac{1}{2}} \right] + \cosh \left[T(1+2s)^{\frac{1}{2}} \right] \right]^{-\frac{1}{2}}$$

$$(B) \mathcal{L}(f_T^C(t))(s) = e^{T/2} \left[(1+2s)^{-\frac{1}{2}} \sinh \left[T(1+2s)^{\frac{1}{2}} \right] + \cosh \left[T(1+2s)^{\frac{1}{2}} \right] \right]^{-\frac{1}{2}},$$

for all complex s such that $\operatorname{Re}(s) \geq (-\frac{1}{2})$.

Proof: Define $w(t)$ by

$$w(t) = \sqrt{2} \left[(t+1)^{\frac{1}{2}} X\left(\frac{1}{2} \log(t+1)\right) - X(0) \right], \quad t \geq 0.$$

Then (Doob, 1942) $w(t)$ is distributed as standard Brownian motion. Solving this equation for $X(\frac{1}{2} \log(t+1))$, substituting into the integral in the equation

$$\begin{aligned} \mathcal{L}(f_T(t))(s) &= E \left[\exp(-s \int_0^T X^2(r) dr) \right] \\ &= E \left[\exp\left(-\frac{s}{2} \int_0^S X^2\left(\frac{1}{2} \log(t+1)\right) \frac{dt}{(t+1)}\right) \right], \end{aligned}$$

where $S = e^{2T} - 1$, we find

$$\mathcal{L}(f_T(t))(s) = E \left[\exp \left[-\frac{s}{4} \int_0^S \frac{[w(t) + \sqrt{2} X(0)]^2}{(t+1)^2} dt \right] \right], \quad (2)$$

$$S = e^{2T} - 1.$$

We now make use of the following theorem of Cameron and Martin (1945): let y be a real number, and let $p(t)$ be a positive, continuous function on $[0,1]$. Let λ_0 be the least characteristic value of the Sturm-Lionville problem

$$f''(t) + \lambda p(t)f(t) = 0 \quad (3)$$

$$f(0) = f'(1) = 0. \quad (4)$$

Suppose $\lambda < \lambda_0$, and f_λ is any solution to (3) satisfying

$$f'_\lambda(1) = 0. \quad (5)$$

Then

$$E\left[\exp\left[\frac{\lambda}{2} \int_0^1 p(t) (w(t) + \sqrt{2} y)^2 dt\right]\right] = \left[\frac{f_\lambda(1)}{f_\lambda(0)}\right]^{\frac{1}{2}} \exp\left[\frac{f'_\lambda(0)}{f_\lambda(0)} y^2\right]. \quad (6)$$

(The factor of 2 arises from the now non-standard normalization $E(w^2(t)) = t/2$ used by Cameron and Martin. Also, the argument of the exponential on the right-hand side of (6) is given by them as

$$y^2 \left[\lambda \int_0^1 p(t) dt + \lambda^2 \int_0^1 \left[\frac{1}{f_\lambda(t)} \int_1^t p(s) f_\lambda(s) ds \right]^2 dt \right].$$

However, using (3) and (5) it is not hard to show that this expression is just $y^2 f'_\lambda(0)/f_\lambda(0)$, as claimed in (6).)

Since we desire to use (6) to evaluate the integral in (2), which has upper limit S , we also need the following scaling result: let λ, y be real numbers, let S, α be positive real numbers, and let $p(t)$ be a positive, continuous function of $t \in [0, \infty)$. Put

$$I(\alpha, \lambda, S, y) = E\left[\exp\left[\frac{\lambda}{2} \int_0^S p(\alpha t) (w(t) + \sqrt{2} y)^2 dt\right]\right]. \quad (7)$$

Then

$$I(\alpha, \lambda, S, y) = I(\alpha S, \lambda S^2, 1, y/\sqrt{S}). \quad (8)$$

To see this, make the change of variables $t = rS$ on the right-hand side of (7) to get

$$I(\alpha, \lambda, S, Y) = E \left[\exp \left[\frac{\lambda S^2}{2} \int_0^1 p(\alpha S r) \left(\frac{w(rS)}{\sqrt{S}} + \frac{\sqrt{2} Y}{\sqrt{S}} \right)^2 dr \right] \right].$$

Since $w(rS)/\sqrt{S}$ is distributed as $w(r)$, (8) follows.

Returning now to (2), we see that putting $p(\alpha t) = (\alpha t + 1)^{-2}$ enables us to write it as

$$\begin{aligned} \varphi(f_T(t))(s) &= \int_{-\infty}^{\infty} E \left[\exp \left[-\frac{s}{4} \int_0^S \frac{[w(t)+y]^2}{(t+1)^2} dt \right] \right] \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dy \\ &= \int_{-\infty}^{\infty} I(1, -s/2, S, Y/\sqrt{2}) \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dy \\ &= \int_{-\infty}^{\infty} I \left(s, -\frac{sS^2}{2}, 1, \frac{Y}{\sqrt{2S}} \right) \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dy, \end{aligned} \quad (9)$$

the last equality arising from (8). But, according to (6), assuming s is real and nonnegative,

$$I \left(s, -\frac{s}{2} S^2, 1, \frac{Y}{\sqrt{2S}} \right) = \left[\frac{g(1)}{g(0)} \right]^{\frac{1}{2}} \exp \left[\frac{g'(0)}{g(0)} \frac{Y^2}{2S} \right], \quad (10)$$

where $g(t)$ is any solution to

$$g''(t) - \left(\frac{s}{2} S^2 \right) (St+1)^{-2} g(t) = 0, \quad t \in [0, 1] \quad (11)$$

satisfying

$$g'(1) = 0. \quad (12)$$

The general solution to (11) is

$$g(t) = c_1 (tS+1)^{\frac{1}{2}+u} + c_2 (tS+1)^{\frac{1}{2}-u}, \quad (13)$$

where

$$u = \frac{1}{2} \sqrt{1+2s}. \quad (14)$$

Since no function of the form (13) satisfies the boundary conditions (4) (because doing so would imply $(S+1)^{2u} = (1-2u)/(1+2u)$, for $u \geq \frac{1}{2}$), the hypothesis $\lambda < \lambda_0$ of the Cameron-Martin theorem is satisfied. The boundary condition (5) requires

$$\frac{c_1}{c_2} = \left[\frac{u-\frac{1}{2}}{u+\frac{1}{2}} \right] (S+1)^{-2u}. \quad (15)$$

Referring to the right-hand side of (10), define A and B by

$$A = \frac{g(1)}{g(0)}, \quad B + \frac{1}{4} = \frac{g'(0)}{g(0)} \frac{1}{2S}. \quad (16)$$

Then (15), together with $S = e^{2T} - 1$, yields

$$A = 2ue^T \left[(u+\frac{1}{2})e^{2uT} + (u-\frac{1}{2})e^{-2uT} \right]^{-1} \quad (17)$$

and

$$B = \frac{u}{2} \left[\frac{(u-\frac{1}{2})e^{-2uT} - (u+\frac{1}{2})e^{2uT}}{(u-\frac{1}{2})e^{-2uT} + (u+\frac{1}{2})e^{2uT}} \right]. \quad (18)$$

Now (9) and (10), together with (16), say that

$$\mathcal{L}(f_T(t))(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{A} \exp\left[(B-\frac{1}{4})y^2\right] dy,$$

or

$$\mathcal{L}(f_T(t))(s) = \left[\frac{2A}{1-4B} \right]^{\frac{1}{2}}.$$

Using (14), (17), and (18) in this formula, together with some straightforward algebraic manipulation, yields part (A) of the theorem.

For the proof of part (B), we note that the time-reversed process $X(T-t)$ is again an Ornstein-Uhlenbeck process, and that

$$Z(T) = \int_0^T X^2(t) dt = \int_0^T X^2(T-t) dt.$$

These observations imply that

$$E\left[\exp(-s \int_0^T X^2(t) dt) | X(T)=y\right] = E\left[\exp(-s \int_0^T X^2(t) dt) | X(0)=y\right]$$

so, by (2), (8), (10) and (16)

$$\mathcal{L}(f_T(t|X(T)=y))(s) = A^{\frac{1}{2}} \exp(2(B+\frac{1}{2})y^2).$$

Putting $y = 0$ yields

$$\mathcal{L}(f_T^C(t))(s) = A^{\frac{1}{2}},$$

and (17), (14), and simple algebra lead to part (B) of the theorem.

While the above argument shows the equalities (A) and (B) only for $s \geq 0$, the analyticity of both sides of these equations establishes their equality for $\text{Res} \geq (-\frac{1}{2})$, completing the proof.

□

INVERSION

We invert the Laplace transforms of theorem 1; our results are stated in terms of distribution functions, rather than densities.

Theorem 2: For all $t, T > 0$,

$$(A) \quad F_T(t) = (2/\pi^3)^{\frac{1}{2}} t^{-3/2} e^{\frac{1}{2}(T-t)} \sum_{n=0}^{\infty} a_n J_n(T, t), \quad (19)$$

$$\text{where} \quad a_n = \frac{(2n-1)!!(2n)!}{2^n n!}, \quad (20)$$

$$J_n(T, t) = \int_0^\infty [v + T(2n + \frac{1}{2})] \exp(-\frac{1}{2t}(v + T(2n + \frac{1}{2}))^2) G_n(v) dv, \quad (21)$$

and

$$G_n(v) = i \int_{r=0}^v \cosh(r) \left[D_{-2n-1}^2(i\sqrt{2(v-r)}) - D_{-2n-1}^2(-i\sqrt{2(v-r)}) \right] dr, \quad (22)$$

D_{-2n-1} being a parabolic cylinder function.

$$(B) \quad F_T^C(t) = \pi^{-1} t^{-3/2} e^{\frac{1}{2}(T-t)} \sum_{n=0}^{\infty} b_n J_n^C(T, t), \quad (23)$$

$$\text{where} \quad b_n = 1/2^n n!, \quad (24)$$

$$J_n^C(T, t) = \int_0^\infty [v + T(2n + \frac{1}{2})] \exp(-\frac{1}{2t}(v + T(2n + \frac{1}{2}))^2) G_n^C(v) dv, \quad (25)$$

and

$$G_n^C(v) = \quad (26)$$

$$\int_{r=0}^v \cosh(r) \int_{u=0}^{v-r} \frac{1}{\sqrt{\pi u(v-r-u)}} \left\{ D_{2n}(2\sqrt{v-r-u}) + D_{2n}(-2\sqrt{v-r-u}) \right\} du dr.$$

Proof: We begin with the proof of (A). Let $f_1(T, t)$ be a function such that

$$\mathcal{L}(f_1(T, t))(s) = e^{T/2} \left[\left[\frac{s^2+1}{2s} \right] \sinh(Ts) + \cosh(Ts) \right]^{-\frac{1}{2}}, \quad (27)$$

and let $F_1(T, t) = \int_0^t f_1(T, r) dr$, so that

$$\mathcal{L}(F_1(T, t))(s) = \frac{1}{s} \mathcal{L}(f_1(T, t)).$$

Then, by algebraic manipulation,

$$\mathcal{L}(F_1(T, t))(s) = \frac{2e^{T/2}}{\sqrt{s}(s+1)} e^{-Ts/2} \left\{ \frac{1}{1 - \left[\left[\frac{s-1}{s+1} \right] e^{-Ts} \right]^2} \right\}^{\frac{1}{2}}.$$

Since $\left| \frac{(s-1)}{(s+1)} e^{-Ts} \right| < 1$ for $s > 0$, for such s we may expand the quantity in curly brackets in a power series to obtain

$$\mathcal{L}(F_1(T, t))(s) = e^{T/2} \sum_{n=0}^{\infty} \frac{a_n}{(2n)!} \left[e^{-Ts(2n+\frac{1}{2})} \frac{2(s-1)^{2n}}{\sqrt{s}(s+1)^{2n+1}} \right], \quad (28)$$

with the a_n 's given by (20) (we use the convention $(-1)!! = 1$). Taking the inverse Laplace transform term-by-term and using (A1) and (A2), we have formally that

$$F_1(T, t) = \frac{i}{\pi} e^{T/2} \sum_{n=0}^{\infty} a_n \chi_{[T(2n+\frac{1}{2}), \infty)}(t) \left\{ D_{-2n-1}^2(i\sqrt{2t-T(4n+1)}) - D_{-2n-1}^2(-i\sqrt{2t-T(4n+1)}) \right\}. \quad (29)$$

To justify (29) rigorously, let $z = iy$, y real, in (A3), obtaining

$$D_{-2n-1}(iy) = \frac{e^{y^2/4}}{(2n)!} \int_0^{\infty} e^{-iyx} e^{-x^2/2} x^{2n} dx,$$

so that

$$|D_{-2n-1}(iy)| \leq \frac{e^{y^2/4}}{(2n)!} \int_0^{\infty} e^{-x^2/2} x^{2n} dx,$$

or

$$|D_{-2n-1}(iy)| \leq e^{y^2/4} \frac{(2n-1)!!}{(2n)!} (\pi/2)^{\frac{1}{2}}, \quad y \text{ real}. \quad (30)$$

Hence, the absolute value of the n th term of the series in (29) is bounded by

$$a_n \pi \left[\frac{(2n-1)!!}{(2n)!} \right]^2 \exp(t - T(2n+\frac{1}{2})),$$

which is easily seen to be e^t times the n th term of a convergent series (e.g., by the ratio test). Thus, by dominated convergence, for $s > 1$ we may take the Laplace transform of the series in (29) term-by-term, arriving at the right side of (28). Since, by (28),

$F_1(T, t)$ and the series in (29) have the same Laplace transform for $s > 1$, they are the same function, as asserted in (29).

By Theorem 1 and (27),

$$\begin{aligned}\mathcal{L}(F_T(t))(s) &= \frac{1}{s} \mathcal{L}(f_T(t))(s) = \left[\left[\frac{2}{s^2-1} \right] \mathcal{L}(f_1(T, t))(s) \right]_{s \rightarrow \sqrt{2s+1}} \\ &= \left[\left[\frac{2s}{s^2-1} \right] \mathcal{L}(F_1(T, t))(s) \right]_{s \rightarrow \sqrt{2s+1}},\end{aligned}$$

where the notation " $s \rightarrow \sqrt{2s+1}$ " indicates that s is to be replaced by $\sqrt{2s+1}$. Using (A4) and (A5), this becomes

$$\mathcal{L}(F_T(t))(s) = \mathcal{L}\left(2 \int_0^t \cosh(u) F_1(T, t-u) du\right)(\sqrt{2s+1}).$$

Applying (A6) and (A7) to the right-hand side, we get

$$\begin{aligned}\mathcal{L}(F_T(t))(s) \\ = \mathcal{L}\left[(2/\pi)^{\frac{1}{2}} t^{-3/2} e^{-t/2} \int_0^\infty v e^{-v^2/2t} \int_0^v \cosh(u) F_1(T, v-u) du dv\right](s).\end{aligned}$$

Using (29), and integrating term-by-term (which may be justified by an argument similar to the one justifying (29)), we see that the inner integral in the above formula is

$$\begin{aligned}\frac{ie^{T/2}}{\pi} \sum_{n=0}^{\infty} a_n \chi_{[T(2n+\frac{1}{2}), \infty)}(v) \int_0^{v-T(2n+\frac{1}{2})} \cosh(u) \left\{ D_{-2n-1}^2(i\sqrt{2(v-u)-T(4n+1)}) \right. \\ \left. - D_{-2n-1}^2(-i\sqrt{2(v-u)-T(4n+1)}) \right\} du,\end{aligned}$$

so that

$$\begin{aligned} \mathcal{L}(F_T(t))(s) = & \mathcal{L} \left[(2/\pi^3)^{\frac{1}{2}} t^{-3/2} e^{\frac{1}{2}(T-t)} \sum_{n=0}^{\infty} a_n \int_{T(2n+\frac{1}{2})}^{\infty} v e^{-v^2/2t} \right. \\ & \times i \int_0^{v-T(2n+\frac{1}{2})} \cosh(u) \left\{ D_{-2n-1}^2(i\sqrt{2(v-u)-T(4n+1)}) \right. \\ & \left. \left. - D_{-2n-1}^2(-i\sqrt{2(v-u)-T(4n+1)}) \right\} du dv \right], \end{aligned}$$

the termwise integration justified, as usual, by (30). Replacing v by $v+T(2n+\frac{1}{2})$ in the integral, and invoking the uniqueness theorem for Laplace transforms, we complete the proof of part (A) of the theorem.

The proof of (B) proceeds, mutatis mutandis, as above. Let $f_1^C(T, t)$ be a function such that

$$\mathcal{L}(f_1^C(T, t)) = e^{T/2} \left[\frac{1}{s} \sinh(Ts) + \cosh(Ts) \right]^{-\frac{1}{2}},$$

and let $F_1^C(T, t) = \int_0^t f_1^C(T, r) dr$.

Then, arguing as before (28), we find

$$\mathcal{L}(F_1^C(T, t))(s) = 2^{\frac{1}{2}} e^{T/2} \sum_{n=0}^{\infty} \frac{a_n (-1)^n}{(2n)!} \left[e^{-Ts(2n+\frac{1}{2})} \frac{(s-1)^n}{\sqrt{s}(s+1)^{n+\frac{1}{2}}} \right]. \quad (31)$$

Then we have formally

$$F_1^C(T, t) = 2^{\frac{1}{2}} e^{T/2} \sum_{n=0}^{\infty} \frac{a_n (-1)^n}{(2n)!} \mathcal{L}^{-1} \left[e^{-Ts(2n+\frac{1}{2})} \frac{(s-1)^n}{\sqrt{s}(s+1)^{n+\frac{1}{2}}} \right]. \quad (32)$$

But by (A5) and (A8)-(A10),

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{(s-1)^n}{\sqrt{s}(s+1)^{n+\frac{1}{2}}} \right] (t) = & \int_0^t \frac{1}{\sqrt{\pi r(t-r)}} \left[D_{2n}(2\sqrt{t-r}) + D_{2n}(-2\sqrt{t-r}) \right] dr \\ & \times (-1)^n / ((2n-1)!! 2\pi^{\frac{1}{2}}), \end{aligned}$$

so, using (A2), we see that (32) is

$$F_1^C(T, t) \quad (33)$$

$$= (2\pi)^{-\frac{1}{2}} e^{T/2} \sum_{n=0}^{\infty} b_n \chi_{[T(2n+\frac{1}{2}), \infty)}(t) \left\{ \int_0^{t-T(2n+\frac{1}{2})} \frac{E_{2n}(t-T(2n+\frac{1}{2})-r)}{\sqrt{\pi r(t-T(2n+\frac{1}{2})-r)}} dr \right\},$$

where, for brevity, we have introduced the notation

$$E_{2n}(x) = D_{2n}(2\sqrt{x}) + D_{2n}(-2\sqrt{x}). \quad (34)$$

To justify (33) rigorously, we have from (A11) that, for z real,

$$|D_{2n}(z)| \leq \pi^{-\frac{1}{2}} 2^{2n+\frac{1}{2}} e^{z^2/4} \int_{-\infty}^{\infty} x^{2n} e^{-2x^2} dx$$

$$= \pi^{-\frac{1}{2}} 2^{2n+\frac{1}{2}} e^{z^2/4} \cdot 2^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} y^{2n} e^{-y^2} dy,$$

which is simply

$$|D_{2n}(z)| \leq (2n-1)!! e^{z^2/4}, \quad z \text{ real}. \quad (35)$$

Using this estimate, and the elementary result

$$\int_0^L \frac{1}{\sqrt{r(L-r)}} dr = \pi, \quad L > 0, \quad (36)$$

we see that the absolute value of the n th term of the series on the right side of (33) is bounded above by

$$b_n \cdot \sqrt{\pi} \cdot 2 \cdot (2n-1)!! \exp(t-T(2n+\frac{1}{2})),$$

which is e^t times the general term of a positive convergent series. Hence, for $s > 1$, we may take the Laplace transform of the right side of (33) termwise, so (33) may be proved as was (29). Using (35) to justify further termwise integrations, we complete the proof of part (B) of the theorem as in the proof of part (A).

ASYMPTOTICS

In this section we study the asymptotic behavior as $T \rightarrow \infty$ of the conditional and unconditional distribution and density functions of $Z(T)$. Our main tool is the following special case of the extended Laplace method (briefly, ELM below) for the asymptotic evaluation of Laplace integrals (Erdelyi, 1956, p. 37). Let $g(v)$ be a function defined on $(0, \infty)$ such that $g(v) \sim bv^{\lambda-1}$ as $v \rightarrow 0$, $\lambda > 0$, and such that, for $k > 0$

$$\int_0^{\infty} g(v) e^{-kTv} dv$$

exists for sufficiently large positive T . Then

$$\int_0^{\infty} g(v) e^{-kTv} dv \sim b\Gamma(\lambda) (kT)^{-\lambda}, \quad T \rightarrow \infty.$$

Theorem 3: Let $f_0(T, t)$ be the probability density function of an inverse Gaussian distribution with parameters $T/2$, $T/2$:

$$f_0(T, t) = (8\pi)^{-\frac{1}{2}} T t^{-3/2} \exp\left[-\frac{t}{2} + \frac{T}{2} - \frac{T^2}{8t}\right], \quad t > 0, \quad (37)$$

and let $F_0(T, t)$ be the corresponding probability distribution function:

$$F_0(T, t) \quad (38)$$

$$= \left(\frac{1}{2}\right) \left[\operatorname{Erfc}\left[-(t/2)^{\frac{1}{2}} + T/(2(2t)^{\frac{1}{2}})\right] + e^T \operatorname{Erfc}\left[(t/2)^{\frac{1}{2}} + T/(2(2t)^{\frac{1}{2}})\right] \right],$$

where $\operatorname{Erfc}(x) = 1 - \operatorname{Erf}(x) = 2\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2} dt$.

Then, for fixed $t > 0$, as $T \rightarrow \infty$,

$$(A) \quad F_T(t) \sim (8t/T)^{\frac{1}{2}} F_0(T, t) \sim 8(\pi)^{-\frac{1}{2}} t T^{-3/2} \exp\left[-\frac{t}{2} + \frac{T}{2} - \frac{T^2}{8t}\right], \quad (39)$$

$$(B) \quad f_T(t) \sim (8t/T)^{\frac{1}{2}} f_0(T, t) \quad (40)$$

$$(C) \quad F_T^C(t) \sim 2^{\frac{1}{2}} F_0(T, t) \sim 4\pi^{-\frac{1}{2}} t^{\frac{1}{2}} T^{-1} \exp\left[-\frac{t}{2} + \frac{T}{2} - \frac{T^2}{8t}\right] \quad (41)$$

$$(D) \quad f_T^C(t) \sim 2^{\frac{1}{2}} f_0(T, t). \quad (42)$$

Note that the final asymptotic equivalences in (A) and (C) follow from

$$F_0(T, t) \sim (8/\pi)^{\frac{1}{2}} t^{\frac{1}{2}} T^{-1} \exp\left[-\frac{t}{2} + \frac{T}{2} - \frac{T^2}{8t}\right], \quad (43)$$

which is a consequence of (A13). The plan of the proof of Theorem 3 is to study via ELM the asymptotic behavior of $J_n(T, t)$, $d/dt(J_n(T, t))$, $J_n^C(T, t)$, and $d/dt(J_n^C(T, t))$ as $T \rightarrow \infty$ (see Lemma 2 below), which in turn depends on the asymptotic behavior of $G_n(v)$ and $G_n^C(v)$ as $v \rightarrow 0$, established in Lemma 1 below. Then, using the uniformity estimates of Lemma 3, we see that the series from Theorem 2 for $F_T(t)$, $f_T(t)$, $F_T^C(t)$, and $f_T^C(t)$ are all asymptotic to their first terms, and the proof is completed by another application of the $n = 0$ case of Lemma 2.

Lemma 1: For all $n \geq 0$,

$$(A) \quad G_n(v) \sim c_n v^{3/2}, \quad v \rightarrow 0, \quad (44)$$

$$\text{where } c_n = 2^{n+3} \pi^{\frac{1}{2}} (2n-1)!! n! / 3((2n)!)^2. \quad (45)$$

$$(B) \quad G_n^C(v) \sim d_n v, \quad v \rightarrow 0, \quad (46)$$

$$\text{where } d_n = 2\pi^{\frac{1}{2}} (-1)^n (2n-1)!! . \quad (47)$$

Proof:

- (A) Since $i \frac{d}{dy}[D_{-2n-1}^2(iy) - D_{-2n-1}^2(-iy)] \rightarrow -4D_{-2n-1}(0)D'_{-2n-1}(0)$ as $y \rightarrow 0$, it follows from (22) that

$$G_n(v) \sim (-2^{7/2}/3)D_{-2n-1}(0)D'_{-2n-1}(0)v^{3/2}, \quad v \rightarrow 0. \quad (48)$$

But it is an easy consequence of (A3) that

$$D_{-2n-1}(0) = (\pi/2)^{\frac{1}{2}}(2n-1)!!/(2n)!,$$

and from (A3) and (A12) it follows that

$$D'_{-2n-1}(0) = -2^n n!/(2n)!,$$

so that (44) and (45) follow from (48).

- (B) By (36), as $v-r \rightarrow 0$,

$$\int_{u=0}^{v-r} \frac{1}{\sqrt{\pi u(v-r-u)}} \left\{ D_{2n}(2\sqrt{v-r-u}) + D_{2n}(-2\sqrt{v-r-u}) \right\} du$$

$$\rightarrow \pi^{\frac{1}{2}} \cdot 2D_{2n}(0) = 2\pi^{\frac{1}{2}}(-1)^n(2n-1)!! ,$$

the last equality resulting from (A11). Equations (46)-(47) then follow from (26).

□

Lemma 2: For fixed $t > 0$ and for all $n \geq 0$, as $T \rightarrow \infty$,

$$(A) \quad J_n(T, t) \sim c_n \Gamma(5/2) t^{5/2} e^{-T^2(2n+\frac{1}{2})^2/2t} (T(2n+\frac{1}{2}))^{-3/2} \quad (49)$$

$$(B) \quad d/dt(J_n(T, t)) \sim 2^{-1} c_n \Gamma(5/2) t^{\frac{1}{2}} e^{-T^2(2n+\frac{1}{2})^2/2t} (T(2n+\frac{1}{2}))^{\frac{1}{2}} \quad (50)$$

$$(C) \quad J_n^C(T, t) \sim d_n t^2 e^{-T^2(2n+\frac{1}{2})^2/2t} (T(2n+\frac{1}{2}))^{-1} \quad (51)$$

$$(D) \quad d/dt(J_n^C(T, t)) \sim 2^{-1} d_n e^{-T^2(2n+\frac{1}{2})^2/2t} (T(2n+\frac{1}{2})), \quad (52)$$

where c_n and d_n are as in Lemma 1.

Proof: From (21) we have

$$J_n(T, t) = e^{-T^2(2n+\frac{1}{2})^2/2t} (I_1 + I_2),$$

where

$$I_1 = \int_0^\infty e^{-T(2n+\frac{1}{2})v/t} e^{-v^2/2t} v G_n(v) dv$$

and

$$I_2 = T(2n+\frac{1}{2}) \int_0^\infty e^{-T(2n+\frac{1}{2})v/t} e^{-v^2/2t} G_n(v) dv.$$

By ELM and Lemma 1(A), as $T \rightarrow \infty$,

$$I_2 \sim c_n \Gamma(5/2) t^{5/2} (T(2n+\frac{1}{2}))^{-3/2}$$

and $I_1 = O(T^{-7/2})$;

thus $I_1 + I_2 \sim I_2$, and part (A) follows.

To prove (B), differentiate (21) with respect to t to obtain $d/dt(J_n(T, t)) = (2t^2)^{-1} \int_0^\infty [v + T(2n+\frac{1}{2})]^3 \exp(-(v + T(2n+\frac{1}{2}))^2/2t) G_n(v) dv$, the differentiation under the integral sign being justified by the uniform convergence of the above integral for t in any compact subset of $(0, \infty)$. Rewriting the above expression, we have

$$\begin{aligned} d/dt(J_n(T, t)) &= (2t^2)^{-1} e^{-T^2(2n+\frac{1}{2})^2/2t} \int_0^\infty [v + T(2n+\frac{1}{2})]^3 e^{-T(2n+\frac{1}{2})v/t} e^{-v^2/2t} G_n(v) dv. \end{aligned}$$

Expanding the cube under the integral sign and investigating the asymptotic behavior of the four terms as $T \rightarrow \infty$ by ELM, we see, as in the proof of (A) above, that the term of the cube with the highest power of T dominates the others. Hence,

$$\begin{aligned} d/dt(J_n(T, t)) &\sim (T(2n+\frac{1}{2}))^3 (2t^2)^{-1} e^{-T^2(2n+\frac{1}{2})^2/2t} \int_0^\infty e^{-T(2n+\frac{1}{2})v/t} e^{-v^2/2t} G_n(v) dv \\ &\sim (T(2n+\frac{1}{2}))^3 (2t^2)^{-1} e^{-T^2(2n+\frac{1}{2})^2/2t} c_n \Gamma(5/2) (T(2n+\frac{1}{2})/t)^{-5/2} \\ &= 2^{-1} c_n \Gamma(5/2) t^{\frac{1}{2}} e^{-T^2(2n+\frac{1}{2})^2/2t} (T(2n+\frac{1}{2}))^{\frac{1}{2}}, \end{aligned}$$

which is (B). The proofs of (C) and (D) are similar to those of (A) and (B), using (25) in place of (21) and Lemma 1(B) instead of Lemma 1(A).

Lemma 3: There exist positive sequences $\{k_n^i\}$, $i = 1, 2$, such that

$$\sum_{n=0}^{\infty} k_n^1 a_n \text{ converges,} \quad (53)$$

$$\sum_{n=0}^{\infty} k_n^2 b_n \text{ converges,} \quad (54)$$

and such that for fixed $t > 0$, for sufficiently large positive T , we have for all $n \geq 0$:

$$(A) \quad \left| \frac{J_n(T, t)}{J_0(T, t)} \right| \leq k_n^1, \quad (55)$$

$$(B) \quad \left| \frac{d/dt(J_1(T, t))}{d/dt(J_0(T, t))} \right| \leq k_n^1, \quad (56)$$

$$(C) \quad \left| \frac{J_n^c(T, t)}{J_0^c(T, t)} \right| \leq k_n^2, \quad (57)$$

$$(D) \quad \left| \frac{d/dt(J_n^c(T, t))}{d/dt(J_0^c(T, t))} \right| \leq k_n^2. \quad (58)$$

Proof: To prove (A), we may begin with (21), square the binomial in the argument of the exponential, and drop the cross term to get

$$|J_n(T, t)| \leq \int_0^{\infty} [v + T(2n + \frac{1}{2})] \exp(-(v^2 + T(2n + \frac{1}{2})^2)/2t) |G_n(v)| dv.$$

But from (22) and (30) it follows that

$$|G_n(v)| \leq \pi \left[\frac{(2n-1)!!}{(2n)!} \right]^2 v e^v \equiv e_n v e^v, \quad (59)$$

so we have

$$|J_n(T, t)| \leq e_n e^{-T^2(2n + \frac{1}{2})^2/2t} [I_1 + T(2n + \frac{1}{2}) I_2],$$

with

$$I_1 = \int_0^{\infty} v^2 e^v e^{-v^2/2t} dv, \quad I_2 = \int_0^{\infty} v e^v e^{-v^2/2t} dv.$$

By Lemma 2(A), for sufficiently large T , we have

$$|J_0(T, t)| > KT^{-3/2}e^{-T^2/8t},$$

where here, as we will do for the remainder of the proof of Lemma 3, we denote by K any expression which may depend on t , but not on T or n . Hence, for such T , we have, for all $n \geq 0$,

$$\left| \frac{J_n(T, t)}{J_0(T, t)} \right| \leq Ke_n T^{3/2} e^{-T^2/2t(4n^2+2n)} (I_1 + T(2n+\frac{1}{2})I_2).$$

Provided $T^2/2t > 1$, this yields

$$\left| \frac{J_n(T, t)}{J_0(T, t)} \right| \leq Ke_n e^{-4n^2} \left[T^{3/2} e^{-T^2 n/t} (I_1 + T(2n+\frac{1}{2})I_2) \right].$$

But, if $T/2t > 5/4$, $T/2t > 1 + 1/4n$, $n \geq 1$, so $T^2 n/t > T(2n+\frac{1}{2})$, $n \geq 1$. Thus

$$\left| \frac{J_n(T, t)}{J_0(T, t)} \right| \leq Ke_n e^{-4n^2} \left[T^{3/2} e^{-T(2n+\frac{1}{2})} (I_1 + T(2n+\frac{1}{2})I_2) \right], \quad n \geq 1.$$

For sufficiently large x , the function xe^{-x} is decreasing, so for sufficiently large T , we have for all $n \geq 1$

$$\left| \frac{J_n(T, t)}{J_0(T, t)} \right| \leq Ke_n e^{-4n^2} \left[T^{3/2} e^{-T/2} (I_1 + (T/2)I_2) \right].$$

Since the last factor above is < 1 for sufficiently large T , part (A) of the Lemma is proved: putting

$$k_0^1 = 1 \text{ and } k_n^1 = Ke_n e^{-4n^2}, \quad n \geq 1,$$

we easily see that $\sum_{n=0}^{\infty} k_n^1 |a_n|$ converges. The proof of part (B) is very similar. To prove (C) and (D), use (26), (35), and (36) to derive the estimate $|G_n^C(v)| \leq 2\pi^{\frac{1}{2}}(2n-1)!!ve^v \equiv \tilde{e}_n ve^v$ to replace

(59) and argue as above, putting

$$k_n^2 = K \tilde{e}_n e^{-4n^2}, \quad n \geq 1,$$

and checking that $\sum_{n=0}^{\infty} k_n^2 |b_n|$ converges. The proof of Lemma 3 is complete. \square

Turning now to the proof of Theorem 3, part (A), we have from Theorem 2(A) that

$$\begin{aligned} \lim_{T \rightarrow \infty} e^{-T/2} F_T(t) / J_0(T, t) &= (2/\pi^3)^{\frac{1}{2}} t^{-3/2} e^{-t/2} \lim_{T \rightarrow \infty} \sum_{n=0}^{\infty} a_n \frac{J_n(T, t)}{J_0(T, t)} \\ &= (2/\pi^3)^{\frac{1}{2}} t^{-3/2} e^{-t/2} \sum_{n=0}^{\infty} a_n \lim_{T \rightarrow \infty} \frac{J_n(T, t)}{J_0(T, t)}, \end{aligned}$$

the interchange of the order of limit and summation being allowed by dominated convergence and Lemma 3(A). But, by Lemma 2(A),

$$\lim_{T \rightarrow \infty} \frac{J_n(T, t)}{J_0(T, t)} = 0, \quad n \geq 1,$$

so, since $a_0 = 1$, we have

$$F_T(t) \sim (2/\pi^3)^{\frac{1}{2}} t^{-3/2} e^{-t/2} e^{T/2} J_0(T, t).$$

Applying Lemma 2(A) again for $n = 0$ proves part (A) of Theorem 3. To prove part (B), we see from Theorem 2(A) that

$$\frac{d}{dt} \left[(2/\pi^3)^{-\frac{1}{2}} t^{3/2} e^{-\frac{1}{2}(T-t)} F_T(t) \right] = \sum_{n=0}^{\infty} a_n \frac{d}{dt} (J_n(T, t)),$$

the termwise differentiation justified by Lemma 3(B). Then

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{d}{dt} \left[(2/\pi^3)^{-\frac{1}{2}} t^{3/2} e^{-t/2} F_T(t) \right] / \frac{d}{dt} (J_0(T, t)) \\ = \lim_{T \rightarrow \infty} \sum_{n=0}^{\infty} a_n \frac{d}{dt} (J_n(T, t)) / \frac{d}{dt} (J_0(T, t)) \end{aligned}$$

$$= \sum_{n=0}^{\infty} a_n \lim_{T \rightarrow \infty} \left[\frac{d}{dt} (J_n(T, t)) / \frac{d}{dt} (J_0(T, t)) \right] = a_0 = 1,$$

where we have used part (B) of Lemma 2 and Lemma 3. Hence

$$(2/\pi^3)^{-\frac{1}{2}} \left[\frac{d}{dt} (t^{3/2} e^{t/2}) F_T(t) + t^{3/2} e^{t/2} f_T(t) \right] \sim e^{T/2} \frac{d}{dt} (J_0(T, t)).$$

But, by part (A) of the theorem and Lemma 2(B) for $n = 0$,

$$F_T(t) = o \left[e^{T/2} \frac{d}{dt} (J_0(T, t)) \right],$$

so we have

$$(2/\pi^3)^{-\frac{1}{2}} t^{3/2} e^{t/2} f_T(t) \sim e^{T/2} \frac{d}{dt} (J_0(T, t)),$$

which, by another application of Lemma 2(B) for $n = 0$, is just part (B) of the theorem. The proof of part (C) is exactly parallel to that of part (A), and the proof of part (D) is exactly parallel to that of part (B); we omit the details.

□

APPENDIX

In this section we list the formulas for Laplace transforms and special functions employed in the text and give references to standard works where they may be found, using, in the style of Gradshteyn and Ryzhik (1980), the following abbreviations:

GR Gradshteyn and Ryzhik (1980)

OB Oberhettinger and Badii (1973)

Thus, "OB 234 (3.65)" refers to formula number 3.65 on page 234 of Oberhettinger and Badii (1973). The formulas are listed in the order in which they are needed in the text. Always in this appendix, $g(s)$ denotes the Laplace transform of $f(t)$:

$$g(s) = \mathcal{L}(f(t))(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

$$(A1) \quad \mathcal{L} \left[\frac{i(2n)!}{\pi} \left\{ D_{-2n-1}^2(i\sqrt{2}t) - D_{-2n-1}^2(-i\sqrt{2}t) \right\} \right] = \frac{2(s-1)^{2n}}{\sqrt{s}(s+1)^{2n+1}}.$$

OB 234 (3.65)

$$(A2) \quad \mathcal{L} \left[\chi_{[b/a, \infty)}(t) f(at-b) \right] = a^{-1} e^{-bs/a} g(s/a), \quad a, b > 0,$$

where $\chi_{[c, \infty)}(t)$ denotes the characteristic function of the

interval $[c, \infty)$: $\chi(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$.

OB 3 (1.7)

$$(A3) \quad D_p(z) = \frac{e^{-z^2/4}}{\Gamma(-p)} \int_0^{\infty} e^{-zx-x^2/2} x^{-p-1} dx, \quad p < 0.$$

GR 1064 (9.241(2))

$$(A4) \quad \mathcal{L}(\cosh(t))(s) = s(s^2-1)^{-1}, \quad \operatorname{Re}(s) > 1$$

OB 84 (9.2)

$$(A5) \quad g_1(s)g_2(s) = \mathcal{L}\left[\int_0^t f_1(u)f_2(t-u)du\right](s)$$

OB 7 (1.31)

$$(A6) \quad g(as+b) = \mathcal{L}\left[a^{-1}e^{-bt/a}f(t/a)\right](s), \quad a > 0$$

OB 207 (1.3)

$$(A7) \quad g(\sqrt{s}) = \mathcal{L}\left[\frac{1}{2}\pi^{-\frac{1}{2}}t^{-3/2}\int_0^\infty ve^{-v^2/4t}f(v)dv\right](s)$$

OB 210 (1.27)

$$(A8) \quad \frac{(s-a)^n}{(s+a)^{n+\frac{1}{2}}} = \Gamma\left(\frac{1}{2}-n\right)\pi^{-1}2^{-n-1}\mathcal{L}\left[t^{-\frac{1}{2}}\left[D_{2n}(2(at)^{\frac{1}{2}}) + D_{2n}(-2(at)^{\frac{1}{2}})\right]\right]$$

OB 182 (17.76)

$\operatorname{Re}(s) > -\operatorname{Re}(a)$

$$(A9) \quad \Gamma\left(\frac{1}{2}-n\right) = (-1)^n 2^n \pi^{\frac{1}{2}} / (2n-1)!!$$

GR 938 (8.399(3))

$$(A10) \quad s^{-\frac{1}{2}} = \mathcal{L}\left[(\pi t)^{-\frac{1}{2}}\right], \quad \operatorname{Re}(s) > 0$$

OB 16 (2.27)

$$(A11) \quad D_p(z) = \pi^{-\frac{1}{2}} 2^{p+\frac{1}{2}} (-i)^p e^{z^2/4} \int_{-\infty}^{\infty} x^p e^{-2x^2} e^{2ixz} dx, \quad p > -1$$

GR 1064 (9.241(1))

$$(A12) \quad D'_p(z) = pD_{p-1}(z) - \frac{1}{2}zD_p(z)$$

GR 1066 (9.247(2))

$$(A13) \quad \operatorname{Erfc}(x) \sim \pi^{-\frac{1}{2}} e^{-x^2} / x, \quad x \rightarrow +\infty$$

GR 931 (8.254)

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